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Hopf solitons and area preserving diffeomorphisms of the sphere

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Abstract

We consider a (3+1)-dimensional local field theory defined on the sphere S^2. The model possesses exact soliton solutions with non trivial Hopf topological charges, and infinite number of local conserved currents. We show that the Poisson bracket algebra of the corresponding charges is isomorphic to that of the area preserving diffeomorphisms of the sphere S^2. We also show that the conserved currents under consideration are the Noether currents associated to the invariance of the Lagrangian under that infinite group of diffeomorphisms. We indicate possible generalizations of the model.

We consider a Lagrangian field theory model defined on four-dimensional Minkowski space-time with coordinates x^μ. The fields of the model form a three-dimensional vector-valued function $\boldsymbol{n}(x)$ satisfying the relation

$$\boldsymbol{n}^2(x) = 1. \tag{1}$$

and so, the target space is the two-dimensional sphere S^2. The Lagrangian density is defined as [1, 2]

$$\mathcal{L} = -\frac{2}{3}\left(\frac{H_{\mu\nu}H^{\mu\nu}}{8}\right)^{3/4}, \tag{2}$$

where the tensor $H_{\mu\nu}$ is defined in terms of the three component unit vector field $\boldsymbol{n} \in S^2$ as

$$H_{\mu\nu} \equiv \boldsymbol{n}\left(\partial_\mu \boldsymbol{n} \times \partial_\nu \boldsymbol{n}\right), \tag{3}$$

where $\partial_\mu \boldsymbol{n}(x) = \partial \boldsymbol{n}(x)/\partial x^\mu$.

The power $3/4$ of $H_{\mu\nu}H^{\mu\nu}$ in (2) is such that the theory circumvents the usual obstacle of the Derrick's scaling argument against existence of stable solitons. We are interested in the boundary condition $\boldsymbol{n} = (0, 0, 1)$ at spatial infinity. This condition compactifies effectively the Euclidean space $\mathbb{R}^3$ to the three-sphere S^3. Accordingly, $\boldsymbol{n}$ becomes a map: $S^3 \to S^2$. Due to the fact that $\pi_3(S^2) = \mathbb{Z}$, the field configurations fall into disjoint classes characterized by the value of the Hopf invariant Q_H.

Using the stereographic projection of S^2, we introduce instead of the vector-valued function $\boldsymbol{n}(x)$ satisfying (1) the scalar complex field $u(x)$ so that

$$\boldsymbol{n} = \frac{1}{1+|u|^2}\left(u + u^*,\, -\mathrm{i}\,(u - u^*),\, 1 - |u|^2\right). \tag{4}$$

For the tensor $H_{\mu\nu}$ one obtains

$$H_{\mu\nu} = \frac{2\,\mathrm{i}}{(1+|u|^2)^2}\left(\partial_\mu u\,\partial_\nu u^* - \partial_\nu u\,\partial_\mu u^*\right). \tag{5}$$

The Euler-Lagrange equations following from (2) are

$$\partial_\mu \mathcal{K}^\mu = 0 \tag{6}$$

and its complex conjugate, and where

$$\mathcal{K}^\mu = \frac{1}{(1+|u|^2)\left(K\partial u^*\right)^{1/4}}\,K^\mu \tag{7}$$

with

$$K^\mu = \left(\partial_\nu u\,\partial^\nu u^*\right)\,\partial^\mu u - \left(\partial_\nu u\,\partial^\nu u\right)\,\partial^\mu u^*. \tag{8}$$

In reference [3] it has been constructed an infinite set of exact soliton solutions with Hopf topological charge $Q_H = -nm$ and masses $M_{m,n} \sim \sqrt{|n|\,|m|\,(|n|+|m|)}$, with m and n being integer numbers. These solitons are similar in some aspects to the solitons of the Skyrme–Faddeev model [4]. They carry the same topological Hopf charge, and they present knot like configurations. The basic difference however is that the static solutions of (2) have an energy invariant under rescaling of the space coordinates. The solitons of the Skyrme–Faddeev model, on the other hand, have a size determined by the balance of the two terms of the Lagrangian with different scaling properties. In addition, the Skyrme–Faddeev model is not an integrable theory and its solutions have been found by numerical methods [5]. The Skyrme–Faddeev theory however, contains a submodel, found in [1], that possesses an infinite number of local conservation laws. That fact may help the development of exact methods in the Skyrme–Faddeev theory. Recently [6] it has been found a similar submodel inside the Skyrme theory [7].

The integrability properties of the model described by the Lagrangian density (2) have been analyzed in reference [1], using the generalized version of the zero curvature condition [8]. It has been shown that such theory possesses an infinite number of conserved currents given by

$$J_G^\mu = \mathrm{i}\left[\mathcal{K}^\mu \frac{\partial G}{\partial u} - \mathcal{K}^{\mu *} \frac{\partial G}{\partial u^*}\right], \tag{9}$$

where G is an arbitrary function of u and u^* but not of their derivatives. We introduced the imaginary unit into relation above to have a real current for a real function G. The conservation of the currents J_G^μ is a consequence of the equations of motion (6), and of the fact that the quantity (8) automatically satisfies the relations

$$K^\mu \partial_\mu u = 0 \qquad\qquad \mathrm{Im}\left(K^\mu \partial_\mu u^*\right) = 0. \tag{10}$$

We calculate in this letter the algebra of the currents J_G^μ under the Poisson bracket. The canonical momentum π conjugated to u is given by

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{u}} = \frac{1}{\left(1+|u|^2\right)^3 \left(K\partial u^*\right)^{1/4}} \left[\, \dot{u}^* \left(\boldsymbol{\nabla} u \boldsymbol{\nabla} u^*\right) - \dot{u} \left(\boldsymbol{\nabla} u^* \boldsymbol{\nabla} u^*\right) \right], \tag{11}$$

where dot denotes the derivative over x^0 and $\boldsymbol{\nabla}$ is the spatial gradient. The expression for the momentum $\pi^* = \partial\mathcal{L}/\partial\dot{u}^*$ can be obtained by complex conjugation. The non-vanishing equal time canonical Poisson bracket relations are

$$\{u(\boldsymbol{x}),\, \pi(\boldsymbol{y})\} = \{u^*(\boldsymbol{x}),\, \pi^*(\boldsymbol{y})\} = \delta(\boldsymbol{x}-\boldsymbol{y}), \tag{12}$$

where $\boldsymbol{x}$ and $\boldsymbol{y}$ stand for the three dimensional space coordinates.

In terms of the momenta, the time component of the currents take the form

$$J_G^0 = \mathrm{i}\left(1+|u|^2\right)^2 \left[\pi \frac{\partial G}{\partial u^*} - \pi^* \frac{\partial G}{\partial u}\right]. \tag{13}$$

A straightforward calculation shows that

$$\{J^0_{G_1}(\boldsymbol{x}),\, J^0_{G_2}(\boldsymbol{y})\} = J^0_{G_{12}}(\boldsymbol{x})\,\delta(\boldsymbol{x}-\boldsymbol{y}), \tag{14}$$

where

$$G_{12} = \mathrm{i}\left(1+|u|^2\right)^2 \left[\frac{\partial G_1}{\partial u}\frac{\partial G_2}{\partial u^*} - \frac{\partial G_1}{\partial u^*}\frac{\partial G_2}{\partial u}\right]. \tag{15}$$

For the corresponding charges

$$\mathcal{Q}_G = \int \mathrm{d}\boldsymbol{x}\, J^0_G(\boldsymbol{x}) \tag{16}$$

we have

$$\{\mathcal{Q}_{G_1},\, \mathcal{Q}_{G_2}\} = \mathcal{Q}_{G_{12}}. \tag{17}$$

Thus, the charges $\mathcal{Q}_G$ form a closed algebra with respect to the Poisson bracket.

It can be shown that the algebra under consideration is actually the Lie algebra of the group of area preserving diffeomorphisms of the two-dimensional sphere. In order to establish that, let us introduce real coordinates u^1 and u^2 connected with the complex coordinate u by $u = u^1 + \mathrm{i}\, u^2$. For the metric tensor components we have

$$g_{ab}(u) = \frac{4}{\left(1+(u^1)^2+(u^2)^2\right)^2}\,\delta_{ab}, \qquad a, b = 1, 2. \tag{18}$$

The area 2-form is defined as

$$A = \frac{1}{2}\sqrt{\det g(u)}\,\epsilon_{ab}\,\mathrm{d}u^a \wedge \mathrm{d}u^b = \frac{2}{\left(1+(u^1)^2+(u^2)^2\right)^2}\,\epsilon_{ab}\mathrm{d}u^a \wedge \mathrm{d}u^b. \tag{19}$$

where $g(u) = \|g_{ab}(u)\|$ and ϵ_{ab} is the totally skew-symmetric symbol normalized by the condition $\epsilon_{12} = 1$. The area form A is invariant under a diffeomorphism Φ of S^2 if

$$\sqrt{\det g(\Phi(u))}\,\det\left\|\frac{\partial \Phi^a(u)}{\partial u^b}\right\| = \sqrt{\det g(u)}. \tag{20}$$

For an infinitesimal diffeomorphism we have

$$\Phi^a(u) = u^a + \varepsilon X^a(u). \tag{21}$$

The infinitesimal version of relation (20) is

$$\partial_a[\sqrt{\det g(u)}\, X^a(u)] = 0. \tag{22}$$

The general solution of this equation can be written as

$$X^a_F(u) = \frac{1}{\sqrt{\det g(u)}}\,\epsilon^{ab}\,\frac{\partial F(u)}{\partial u^b}, \tag{23}$$

where F is an arbitrary function of the coordinates u^1 and u^2.

Recall that the commutators of the vector fields describing infinitesimal transformations of some Lie transformation group reproduce the Lie algebra of the Lie group under consideration. In general we have

$$[X_{F_1}, X_{F_2}] = X_{F_{12}}. \tag{24}$$

and calculations show that

$$F_{12}(u) = -\frac{1}{\sqrt{\det g(u)}}\, \epsilon^{ab}\, \frac{\partial F_1(u)}{\partial u^a}\frac{\partial F_2(u)}{\partial u^b}\,. \tag{25}$$

Using the complex coordinate u we come to

$$F_{12} = \frac{\mathrm{i}}{2}\left(1+|u|^2\right)^2\left[\frac{\partial F_1}{\partial u}\frac{\partial F_2}{\partial u^*} - \frac{\partial F_1}{\partial u^*}\frac{\partial F_2}{\partial u}\right] \tag{26}$$

and so, the algebras (17) and (24) are isomorphic under the correspondence $\mathcal{Q}_{F/2} \leftrightarrow X_F$. Therefore, the algebra of the charges $\mathcal{Q}_G$ is indeed the Lie algebra of the group of area preserving diffeomorphisms of S^2.

Certainly, the established isomorphism of the Lie algebras is not accidental. As we now show, it is a consequence of the fact that the Lagrangian is invariant under the area preserving diffeomorphisms of S^2, and consequently (9) are the corresponding Noether currents. Notice that using the complex coordinate u, we obtain the following expression for the area form

$$A = \frac{2\,\mathrm{i}}{\left(1+|u|^2\right)^2}\,\mathrm{d}u \wedge \mathrm{d}u^*. \tag{27}$$

Using this expression, we see that the 2-form

$$\begin{aligned}&\frac{1}{2}\,H_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \\ &\quad = \frac{\mathrm{i}}{\left(1+|u|^2\right)^2}\left(\partial_\mu u\,\partial_\nu u^* - \partial_\nu u\,\partial_\mu u^*\right)\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \frac{2\,\mathrm{i}}{\left(1+|u|^2\right)^2}\,\mathrm{d}u \wedge \mathrm{d}u^*\end{aligned} \tag{28}$$

is the pull-back of the area form on S^2. Therefore this 2-form is invariant with respect to the field transformations induced by the area preserving diffeomorphism of S^2 and any Lagrangian constructed from $H_{\mu\nu}$ is invariant under those transformations. It then follows that the currents (9) are in fact the Noether currents associated with the area preserving diffeomorphisms.

In general one can consider the m-dimensional Minkowski space-time and take as the target manifold of a model an arbitrary n-dimensional Riemannian manifold N with $n \leq m$. The vector fields describing infinitesimal volume preserving diffeomorphisms of N are given by

$$X_F^a(u) = \frac{1}{(n-2)!}(\det g(u))^{-1/2}\epsilon^{aa_1\ldots a_{n-1}}\partial_{a_1}F_{a_2\ldots a_{n-1}}(u), \tag{29}$$

where $F_{a_1 \ldots a_{n-2}}$ are the components of an $(n-2)$-form F on N. The tensor

$$H_{\mu_1 \ldots \mu_n} = \frac{1}{n!} \sqrt{\det g(u)}\, \epsilon_{a_1 \ldots a_n}\, \partial_{\mu_1} u^{a_1} \ldots \partial_{\mu_n} u^{a_n} \tag{30}$$

is invariant with respect to the field transformation induced by the volume preserving diffeomorphism of the manifold N. Therefore, if we construct the density of the Lagrangian as an arbitrary function of

$$H_{\mu_1 \ldots \mu_n} H^{\mu_1 \ldots \mu_n} = \frac{1}{n!} \det g(u) \det \|\partial_\mu u^a \partial^\mu u^b\| \tag{31}$$

we will come to a Lorentz invariant model which is also invariant with respect to the volume preserving diffeomorphisms of the target manifold N.

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